HARD 1

3 4 5 6 7 8 28 30 35

Split the numbers above into three groups of three numbers each, so that the product of the numbers in each group is equal.

Answer: (3, 8, 35), (4, 7, 30) and (5, 6, 28).

Solution: The product of all the numbers is 2^9 x 3^3 x 5^3 x 7^3. That means the product of each group must be 2^3 x 3 x 5 x 7 = 840. Now, 840/30 = 28, so whichever group contains the number 30, must also contain two numbers multiplying to 28; the only possibility is 4 x 7 and so one group is (4, 7, 30). Similarly, the group containing 28 must also contain 5 and 6, and then the final group is (3, 8, 35).

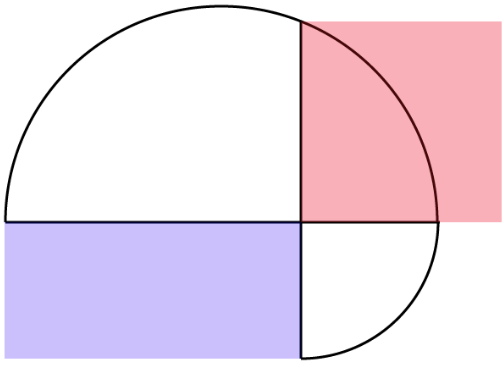
HARD 2

Lucky Leonhard is playing a casino game. He places his bet and then rolls a pair of dice: if the dice sum to 7, 8 or 9 then Leonhard wins the amount of his bet, and otherwise Leonhard loses his money. Leonhard begins with $100 and decides to bet $50 at time, until he has either won $100 or has gone broke. What are the chances that Leonhard leaves the casino a winner?

Answer: 25/74.

Solution: The chance of Leonhard winning on a given roll is 15/36 = 5/12. There is then a 5/12 x 5/12 = 25/144 chance that Leonhard will win his first two rolls (and leave the casino). Otherwise, the only way Leonhard can still have money after the first two rolls is if he won one roll and lost the other, and the chance of that is 2 x 5/12 x 7/12 = 70/144; in this case Leonhard is back where he started. If we let *D* stand for Leonhard's chances of doubling his money, then we must have D = 25/144 + (70/144)D. Solving for D, it follows that D = 25/74.

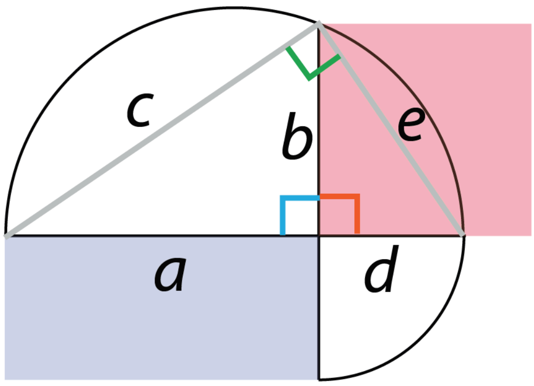
HARD 3



In the above picture the top curve is a semicircle and the bottom curve is a quarter circle. Which has greater area, the red square or the blue rectangle?

Answer: The areas are equal.

Solution: There are three right-angled triangles hiding in the picture.



They tell us that

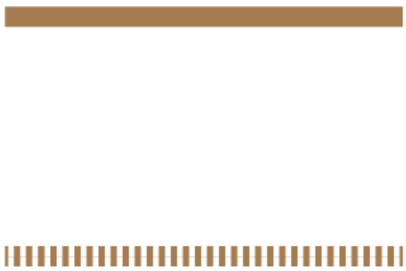
$a^2 + b^2 = c^2$

$b^2 + d^2 = e^2$

$c^2 + e^2 = (a + d)^2$

Adding the three equations and cancelling, it follows that $2b^2 = 2ad$, and so the rectangle and square have the same area.

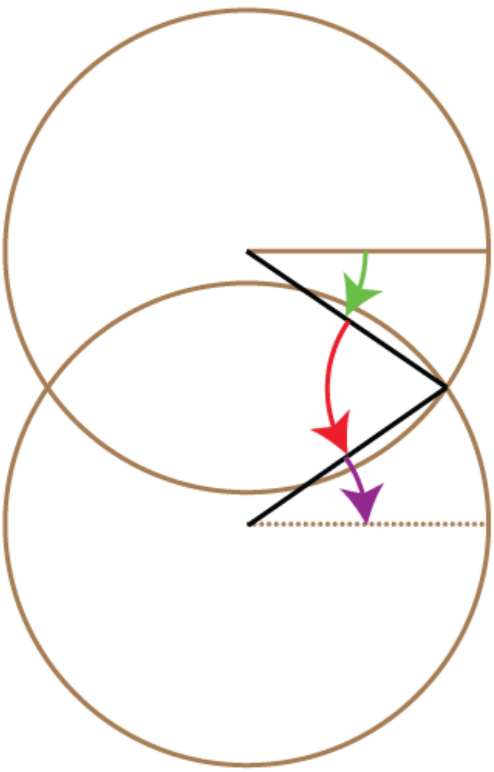
HARD 4



Felix wants to move the pictured log into the position indicated by the dotted line. The log is so heavy that Felix can only lift up one end and rotate the log around the other end. How many such moves does Felix require to accomplish his task.

Answer: 3 moves

Solution:



HARD 5

In a 100 meter race, Jacob can beat Johann by 5 meters, and Johann can beat Nicolaus by 10 meters. By how much can Jacob beat Nicolaus?

Answer: 14.5 meters

Solution: Suppose Jacob runs at speed V. Then Johann runs at speed $19V/20$. Since Nicolaus runs at 9/10 of the speed of Jacob, he runs at speed $171V/200$. So, when Jacob runs 100 meters, Nicolaus will have run 85.5 meters.

HARD 6

$$3 + 4 + 5 = 12$$

$$6 + 7 + 8 = 21$$

The above are two pretty equations: six consecutive numbers with the sum of the first three giving a two-digit number and the sum of the second three resulting in the same number but with the digits reversed. Are there any other such examples?

Answer: The other possibilities are the sequences starting with 14 and 25.

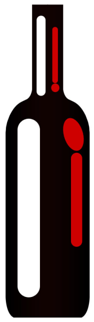
Solution: Suppose the sequence of numbers begins with X and the sum of the first three numbers has digits AB. Then we have the two equations

$X + (X + 1) + (X + 2) = 10A + B$

$(X + 3) + (X + 4) + (X + 5) = 10B + A$

Subtracting the first equation from the second, it follows that $9 = 9B - 9A$, and so $B = A + 1$. That means the possible sums of our first three numbers are 12, 23, 34, 45, 56, 67, 78, and 89. However, the left hand side of the first equation = $3X + 3$ is divisible by 3. That means the only possible sums are 12, 45 and 78. The two new solutions are the sequences starting with 14 (so 14 + 15 + 16 = 45 and 17 + 18 + 19 = 54), and 25.

HARD 7



René has a 16 ounce bottle of wine. On the first day he drinks 1 ounce and then fills the bottle back up with water. On the second day he drinks 2 ounces of the mixture in the bottle and then again fills the bottle with water. The next day he drinks 3 ounces, and so on, until on the 16th day he drinks the full contents of the bottle. How much water did René drink?

Answer: 120 ounces.

Solution: In the end, René drank all the water he poured into the bottle, amounting to 1 + 2 + 3 + … + 15 = 120 ounces.

HARD 8

$$100^2 - 99^2 + 98^2 - 97^2 + \ldots + 2^2 - 1^2$$

What is the above sum?

Answer: 5050.

Solution: Notice that $100^2 - 99^2 = (100 - 99)(100 + 99) = 100 + 99$, similarly $98^2 - 97^2 = 98 + 97$, and so on. So, our whole sum is

100 + 99 + 98 + … + 3 + 2 + 1.

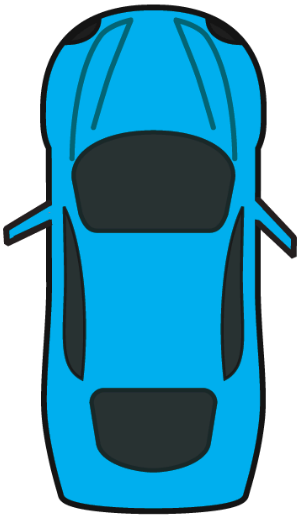
There is then a famous trick for calcuating this sum:

S = 100 + 99 + 98 + … + 3 + 2 + 1

S = 1 + 2 + 3 + … + 98 + 99 + 100

So 2S = 101 + 101 + … + 101 = 100 x 101 = 10100, and then S = 5050.

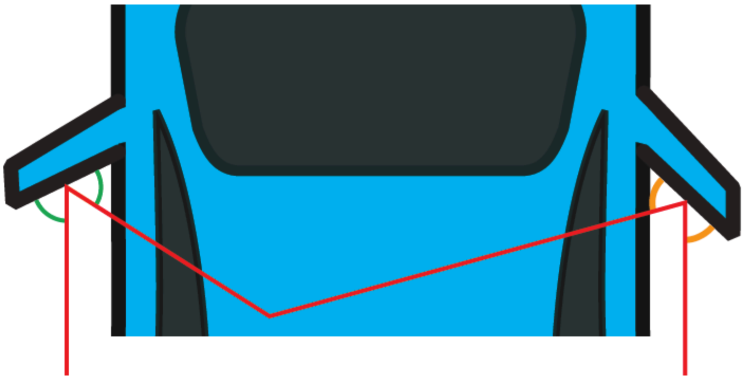
HARD 9



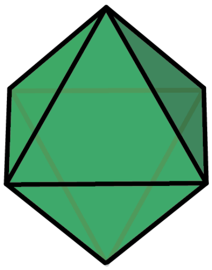
Is the steering wheel on the left or right side of the car?

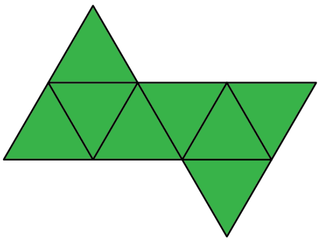
Answer: The left side.

Solution: The slight difference in orientation of the side mirrors indicates where the driver's seat must be:



HARD 10





Above is pictured an octahedron ("diamond") together with one if its *nets*: a way of cutting open the octahedron along its edges so that it is still all connected but can lie flat on the table. The octahedron has 11 nets in total: can you find them all? (Reflected and rotated nets are not considered to be different.)

Answer: If you have 11 different nets then you're done!

Solution: The eleven possible nets are pictured below.